Name:…………………………………………………………………………………….

Student Number: …………………………………………………………………..

Test 6 on WPPH16001.2017-2018
“Electricity and Magnetism”

Content: 12 pages (including this cover page)

Wednesday 20 June 2018; MartiniPlaza L Springerlaan 2, 9:00-12:00

- Write your full name and student number
- Write your answers in the designated area; if you use **extra sheets, indicate this clearly!**
- Reversed sides of each page are left blank intentionally and could be used for draft answers
- Read the questions carefully
- Please, do not use a pencil
- Books, notes, phones, smartphones and tablets are not allowed. Calculators and dictionaries are allowed.

Exam drafted by (name first examiner) Maxim S. Pchenitchnikov
Exam reviewed by (name second examiner) Steven Hoekstra

The weighting of the questions and the grading scheme (total number of points, and mention the number of points at each (sub)question).
Grade = 1 + 9 x (score/max score).

For administrative purposes; do NOT fill the table

	Maximum points	Points scored
Question 1	6	
Question 2	16	
Question 3	13	
Question 4	13	
Question 5	14	
Total	**62**	

Final mark: _________

Question 1 (6 points)

Consider a parallel-plate capacitor immersed in sea water and driven by a voltage $V = V_0 cos(2\pi \nu t)$. Sea water has permittivity $\epsilon = 81\epsilon_0$, permeability $\mu = \mu_0$, and resistivity $\rho = 0.23\ \Omega \cdot m$ at the driving frequency $\nu = 4 \cdot 10^8 Hz$.
What is the ratio (provide a value!) of the amplitudes of conduction current to displacement current?

<u>Answer to Question 1</u> (Griffiths, Problem 7.40) **(6 points)**

Conduction current $J_c = \sigma E = \frac{1}{\rho}E = \frac{1}{\rho}\frac{V}{d} = \frac{1}{\rho d}V_0 cos(2\pi \nu t)$ (2 points)

Displacement current:

$J_d = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}(\epsilon E) = \epsilon \frac{\partial}{\partial t}\left[\frac{V_0 cos(2\pi \nu t)}{d}\right] = \frac{\epsilon}{d} 2\pi \nu V_0 \left(-sin(2\pi \nu t)\right)$ (2 points)

Ratio of their amplitudes: $\frac{A_c}{A_d} = \frac{V_0}{\rho d}\frac{d}{2\pi \nu \epsilon V_0} = \frac{1}{2\pi \nu \epsilon \rho} = (2\pi \nu \epsilon \rho)^{-1}$ (1 point)

$\frac{1}{2\pi \nu \epsilon \rho} = \frac{1}{2\pi\ 4\cdot 10^8 \cdot 81 \cdot 8.85 \cdot 10^{-12} \cdot 0.23} = 2.4$ (1 point)

Question 2 (16 points)

An ideal linear isotropic conductor with conductivity σ occupies the space $z > 0$.

A. Write Maxwell's equations and Ohm's law that govern the electromagnetic field inside the conductor. Consider $\rho_f = 0$ inside the conductor. (2 points)

B. Derive a modified wave equation $\nabla^2 \mathbf{E} = \mu\sigma \frac{\partial \mathbf{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$ for the electromagnetic field inside the conductor (Hint: use the very same approach as you did in the lecture, i.e. apply the *curl* operator to the respective Maxwell's equation) (4 points)

C. Using a trial plane-wave solution $\tilde{\mathbf{E}}(z, t) = \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z - \omega t)}$, find both the real $Re[\tilde{k}]$ and imaginary $Im[\tilde{k}]$ parts of the now-complex wavenumber $\tilde{k} = Re[\tilde{k}] + iIm[\tilde{k}]$. (Hint: use the approximation of an ideal conductor $\sigma/\epsilon\omega \gg 1$ and the relation $\sqrt{i} = (1 + i)/\sqrt{2}$). (6 points)

D. Find the distance d it takes to reduce the amplitude of the electric field by a factor of $1/e$ upon the propagation in the conductor (the so-called *skin depth*). Express this distance in terms of the wavelength λ *in the conductor*. (4 points)

Answer to question 2 (Problem 9.20b from Griffiths and p.413) 16 points

A. (2 points)

$$\nabla \cdot \mathbf{E} = 0 \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{B} = \mu \mathbf{J} + \mu\epsilon \frac{\partial \mathbf{E}}{\partial t}$ (from the formula sheet, 1 point)

$\mathbf{J} = \sigma \mathbf{E}$ (from the formula sheet, 1 point)

B. (4 points)

$\nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}\left(\mu\sigma\mathbf{E} + \mu\epsilon\frac{\partial \mathbf{E}}{\partial t}\right) = -\mu\sigma\frac{\partial \mathbf{E}}{\partial t} - \mu\epsilon\frac{\partial^2 \boldsymbol{E}}{\partial t^2}$ (2 points)

$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}$ (1 point)

The wave equation: $\nabla^2 \mathbf{E} = \mu\sigma \frac{\partial \mathbf{E}}{\partial t} + \mu\epsilon\frac{\partial^2 \boldsymbol{E}}{\partial t^2}$ (1 point)

C. (6 points)

$-\tilde{k}^2 \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z - \omega t)} = -i\omega\mu\sigma\tilde{\mathbf{E}}_0 e^{i(\tilde{k}z-\omega t)} - \mu\epsilon\omega^2\tilde{\mathbf{E}}_0 e^{i(\tilde{k}z-\omega t)}$ (2 points)

$\tilde{k}^2 = i\mu\sigma\omega + \mu\epsilon\omega^2 = i\mu\sigma\omega\left(1 - i\frac{\epsilon\omega}{\sigma}\right) \cong i\mu\sigma\omega$ (2 points)

$$\tilde{k} = \sqrt{i\mu\sigma\omega} = \sqrt{\frac{\mu\sigma\omega}{2}}(1+i)$$

$\tilde{k} = Re[\tilde{k}] + iIm[\tilde{k}]; \;\; Re[\tilde{k}] = \sqrt{\frac{\mu\sigma\omega}{2}}; \; Im[\tilde{k}] = \sqrt{\frac{\mu\sigma\omega}{2}}$ (2 points)

D. (4 points)

$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{-Im[\tilde{k}]z} e^{i(Re[\tilde{k}]z - \omega t)}$, (1 point)

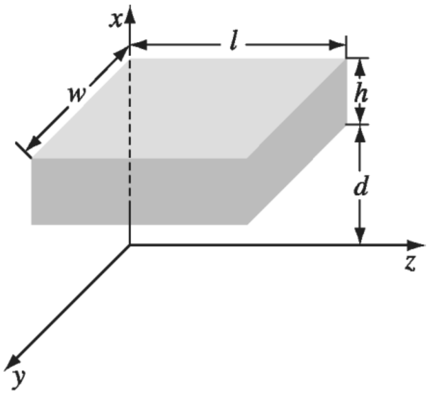
i.e. $d = 1/Im[\tilde{k}] = \sqrt{2/(\mu\sigma\omega)}$ (1 points)

As before, $\lambda = 2\pi/Re[\tilde{k}]$; (1 point)

$(Re[\tilde{k}] = Im[\tilde{k}])$

i.e. $\lambda = 2\pi d; \, d = \lambda/2\pi$ (1 point)

Question 3 (13 points)



Consider a rectangular box of length *l*, width *w*, and height *h*, situated a distance *d* above the *yz* plane (Figure).

The scalar and vector potentials are given as follows:

$V = 0$

$$\mathbf{A} = \begin{cases} \frac{\mu_0 k}{4c}(ct - |x|)^2\hat{\mathbf{z}}, for\ |x| < ct \\ \mathbf{0}, \qquad\qquad\qquad for\ |x| > ct \end{cases}$$

where *k* is a constant, $c = 1/\sqrt{\mu_0 \epsilon_0}$ (as usual), and $\mu = \mu_0, \epsilon = \epsilon_0$ everywhere.

A. Find the electric field. (2 points)

B. Find the magnetic field. (5 points)

C. Find the energy $W(t_1)$ in the box at time $t_1 = d/c$. (1 point)

D. Find the energy $W(t_2)$ in the box at time $t_2 = (d + h)/c$. (5 points)

Answer to Question 3 (Griffiths Example 10.1 and Problem 10.2) 13 points

A. (2 points)

$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -\frac{\partial \mathbf{A}}{\partial t}$ (1 point)

$= -\frac{\mu_0 k}{2}(ct - |x|)\ \hat{\mathbf{z}}, for\ |x| < ct$ (1 point)

$\mathbf{E} = 0, for\ |x| > ct$

B. (5 points)

$\mathbf{B} = \nabla \times \mathbf{A}$ (1 point)

$= \frac{\mu_0 k}{4c}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & \frac{\partial}{\partial z} \\ 0 & 0 & (ct-|x|)^2 \end{vmatrix} = -\frac{\mu_0 k}{4c}\frac{\partial}{\partial x}(ct-|x|)^2\,\hat{\mathbf{y}}$ (2 points)

$= \begin{cases} +\frac{\mu_0 k}{2c}(ct-|x|)\ \hat{\mathbf{y}}, for\ x > 0 \\ -\frac{\mu_0 k}{2c}(ct-|x|)\ \hat{\mathbf{y}}, for\ x < 0 \end{cases}, for\ |x| < ct$ (2 points)

$\mathbf{B} = 0, for\ |x| > ct$

Note: no point subtracted if the sign in the front is not ± but only +

C. (1 point)

$$W = \frac{1}{2}\int\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau.$$

At $t_1 = d/c,\ x \geq d = ct_1$, so $\mathbf{E} = 0,\ \mathbf{B} = 0$, and hence $\boxed{W(t_1) = 0.}$

D. (5 points)

At $T_2 = (d+h)/c,\ ct_2 = d+h$:

$$\mathbf{E} = -\frac{\mu_0 k}{2}(d+h-x)\,\hat{\mathbf{z}},\quad \mathbf{B} = \frac{1}{c}\frac{\mu_0 k}{2}(d+h-x)\,\hat{\mathbf{y}},\ \text{so } B^2 = \frac{1}{c^2}E^2, \text{ and}$$

$$\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) = \epsilon_0\left(E^2 + \frac{1}{\mu_0\epsilon_0}\frac{1}{c^2}E^2\right) = 2\epsilon_0 E^2$$

$$W(t_2) = \frac{1}{2}(2\epsilon_0)\frac{\mu_0^2 k^2}{4}\int_d^{(d+h)}(d+h-x)^2\,dx\,(lw) = \frac{\epsilon_0\mu_0^2 k^2 lw}{4}\left[-\frac{(d+h-x)^3}{3}\right]_d^{d+h} = \boxed{\frac{\epsilon_0\mu_0^2 k^2 lwh^3}{12}.}$$

Question 4 (13 points)

A. Calculate the total power radiated by an oscillating electric dipole. (Tip: you might find useful the surface element $da = r^2 sin\theta \, d\theta \, d\varphi$ and the following integral $\int sin^3\theta \, d\theta = \frac{cos^3\theta}{3} - cos\theta$) (3 points)

B. Find the "radiation resistance" R_{rad} of the wire joining the two ends of the radiative dipole. This is the resistance that would give the same average power loss-to-heat as the oscillating dipole in *fact* puts out in the form of radiation. (Tip: use Joule's law and the energy conservation law) (5 points)

C. Express your result in the size of the dipole d and the emitted wavelength λ; re-calculate all constants into a single number. (3 points)

D. For the wires in an ordinary radio (say, $d = 1 \ cm$ and $\lambda = 1 \ m$), calculate and compare the values of the radiative resistance R_{rad} and the total resistance R. For the latter, consider a copper (resistivity $\rho = 2 \ 10^{-8} \ \Omega \ m$) wire of diameter $D = 1 \ mm$ (2 points)

Answer to Question 4 (Griffiths, p.471 and Problem 11.3) 13 points

A. (3 points)

$\langle S\rangle = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\left(\frac{sin^2\theta}{r^2}\right)\hat{\mathrm{r}}$ (from formula sheet)

The total power radiated: $\langle P\rangle = \int \langle \mathbf{S}\rangle \cdot d\mathbf{a} = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\int \frac{sin^2\theta}{r^2} r^2 sin\theta\, d\theta d\varphi$ (1 point)

$= \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\int_0^{2\pi} d\varphi \int_0^{\pi} sin^3\theta\, d\theta = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} 2\pi \left(\frac{cos^3\theta}{3} - cos\theta\right)\Big|_0^{\pi} = \frac{\mu_0 p_0^2 \omega^4}{16\pi c}\frac{4}{3} = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$ (2 point)

B. (5 points)

$I = \frac{dq}{dt} = -q_0\omega sin(\omega t)$ (1 point)

$P = I^2 R = q_0^2\omega^2 sin^2(\omega t) R$ (1 point)

Averaged power $\langle P\rangle = \frac{1}{2} q_0^2 \omega^2 R$ (1 point)

$$\frac{1}{2} q_0^2 \omega^2 R = \frac{\mu_0 d^2 q_0^2 \omega^4}{12\pi c}$$

$R = \frac{\mu_0 d^2 \omega^2}{6\pi c}$ (2 points)

C. (3 points)

$\omega = \frac{2\pi c}{\lambda}$ (1 point)

$$R_{rad} = \frac{\mu_0 d^2 4\pi^2 c^2}{6\pi c \lambda^2} = \frac{2}{3}\pi\mu_0 c\left(\frac{d}{\lambda}\right)^2 = \frac{2}{3}\pi\,(4\pi \times 10^{-7})(3\times 10^8)\left(\frac{d}{\lambda}\right)^2 \cong 790\left(\frac{d}{\lambda}\right)^2\ [\Omega]$$

(2 points)

D. (2 points)

$R_{rad} = 790\left(\frac{0.01}{1}\right)^2 = 0.079\ \Omega;\ R = \rho\frac{d}{(\pi D^2/4)} = 2\ 10^{-8}\ \frac{0.01}{(\pi 10^{-6}/4)} = 2.5\ 10^{-4}\ \Omega$

R is negligible compared to R_{rad}. (2 points)

Question 5 (14 points)

Albert Einstein wondered at age 16 what the fields **E** and **B** are if you would travel together with the plain electromagnetic wave. He deduced it at age 26 when he applied the Lorentz transformation to the electromagnetic field. Following Einstein, find the electromagnetic field if you travel along the electromagnetic wave.

A. Write down the electric and magnetic fields of a sinusoidal electromagnetic plane wave of angular frequency ω, amplitude E_0 and phase $\delta = 0$, and polarized into the y-direction, which is travelling into the x-direction through the vacuum. (2 points)

B. The same wave is observed from an inertial frame $\bar{S}$, moving with the speed v in the x-direction. Find all (i.e. x, y, z) components the electric and magnetic fields in $\bar{S}$ using the following substitution $\alpha \equiv \gamma\left(1 - \frac{v}{c}\right)$ (6 points)

C. What is the ratio of the intensity $\bar{I}$ in $\bar{S}$ to the intensity I in S? (3 points)

D. Now v approaches c. What are the amplitudes and intensity of the electromagnetic wave? (Tip: be careful as when $v \to c, \gamma \to \infty$ so that first express α via v and c) (3 points)

Answers to question 5 (Problem 12.48) 14 points

A. (2 points)

$\mathbf{E}(x,t) = E_0 e^{i(kx-\omega t)}\hat{\mathbf{y}}$ (1 points)

$\mathbf{B}(x,t) = \frac{1}{c} E_0 e^{i(kx-\omega t)}\hat{\mathbf{z}}$ (1 points)

Wrong polarization in either field results in -1/2 point
Wrong scaling in the magnetic field results in -1/2 point

B. (6 points)

$\bar{E}_x = E_x = 0;\ \bar{E}_z = \gamma\left(E_z + vB_y\right) = 0$ (1 point)

$\bar{B}_x = B_x = 0;\ \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right) = 0$ (1 point)

$\bar{E}_y = \gamma\left(E_y - vB_z\right) = \gamma E_0\left(1 - \frac{v}{c}\right)e^{i(kx-\omega t)} = \alpha E_0 e^{i(kx-\omega t)}$ (2 points)

where $\alpha = \gamma\left(1 - \frac{v}{c}\right)$

$\bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right) = \gamma E_0\left(\frac{1}{c} - \frac{v}{c^2}\right)e^{i(kx-\omega t)} = \frac{\alpha}{c}E_0 e^{i(kx-\omega t)}$ (2 points)

C. (3 points)

$I = \frac{1}{2}c\epsilon_0 E_0^2;\ \bar{I} = \frac{1}{2}c\epsilon_0 \bar{E}_0^2$ (from formula sheet, 1 point)

$\frac{\bar{I}}{I} = \frac{\bar{E}_0^2}{E_0^2} = \alpha^2 = \frac{\left(1-\frac{v}{c}\right)}{\left(1+\frac{v}{c}\right)}$ (2 points)

D. (3 points)

$$\alpha = \gamma\left(1 - \frac{v}{c}\right) = \frac{\left(1-\frac{v}{c}\right)}{\left(1 - v^2/c^2\right)^{1/2}} = \frac{\left(1-\frac{v}{c}\right)}{\left(1-\frac{v}{c}\right)^{1/2}\left(1+\frac{v}{c}\right)^{1/2}} = \frac{\left(1-\frac{v}{c}\right)^{1/2}}{\left(1+\frac{v}{c}\right)^{1/2}}$$

As $\alpha \to 0$ with $v \to c$, $\bar{E}_0 = 0$ and $\bar{I} = 0$ (3 points)

Maximum S. Pchenitchnikov

M. Pshenis

12 June 2018

Steven Hoekstra

12 June 2018